High dimensional Schwarzian derivatives and Painlevé integrable models

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Abstract

Because of all the known integrable models possess Schwarzian forms with Möbious transformation invariance, it may be one of the best way to find new integrable models starting from some suitable Möbious transformation invariant equations. In this paper, the truncated Painlevé analysis is used to find high dimensional Schwarzian derivatives. Especially, a three dimensional Schwarzian derivative is obtained explicitly. The higher dimensional higher order Schwarzian derivatives which are invariant under the Möbious transformations may be expressed by means of the lower dimensional lower order ones. All the known Schwarzian derivatives can be used to construct high dimensional Painlevé integrable models.

1 Introduction

Soliton as the most basic excitation of the integrable models has been widely applied in natural science[1]. However, most of the present studies of the soliton theory and soliton applications are restricted in (1+1)- and (2+1)-dimensions. The essential reason is the lack of known higher dimensional integrable systems.

Actually, to our knowledge, almost all the known integrable (1+1)- and (2+1)-dimensional models possess Schwarz invariant forms which are invariant under the Möbious transformation (conformal invariance)[2, 3, 5]. Starting from the conformal invariance of a known integrable model, one may obtain various other interesting integrable properties. For instance, the conformal invariance of the well known Schwarzian Korteweg de-Vries (SKdV) equation is related to the infinitely many symmetries of the usual KdV equation[6]. The Darboux transformation[7] and the Bäcklund transformation[2] can be obtained from the conformal invariance. In Ref. [8],

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it is pointed out that every (1+1)-dimensional Möbious invariant model possesses a second order Lax pair. The flow equation related to the conformal invariance of the SKdV equation is linked with some types of (1+1)-dimensional and (2+1)-dimensional sinh-Gordon (ShG) equations and Mikhailov-Dodd-Bullough (MDB) equations[9].

According to the above marvelous properties of the Schwarzian forms, in [5], one of the present authors (Lou) proposed that starting from a conformal invariant form may be one of the best ways to find integrable models especially in high dimensions. Some types of quite general Schwarzian equations are proved to be Painlevé integrable. In [10], Conte's conformal invariant Painlevé analysis[11] is extended to obtain high dimensional Painlevé integrable Schwarzian equations systematically. Some types of physically important high dimensional nonintegrable models can be solved approximately via some high dimensional Painlevé integrable Schwarzian equations[12].

In all our previous papers, we used only one dimensional Schwarzian derivative to find some possible new integrable models especially in high dimensions. Now two important and interesting questions are: Are there any high dimensional Schwarzian derivatives (high dimensional derivatives with Möbious transformation invariance) and can we obtain some new types of integrable models by using high dimensional Schwarzian derivatives if there exist?

In the next section, we establish a possible way to obtain explicit high dimensional Schwarzian derivatives from the Painlevé analysis of partial differential equations (PDEs). In section 3, we use the obtained (2+1)-dimensional Schwarzian derivatives to construct some generalized (2+1)- and (3+1)-dimensional integrable models. The last section is a short summary and discussion.

2 Obtain high dimensional Schwarzian derivatives via truncated Painlevé analysis

It is known that the truncated Painlevé analysis of the integrable models will lead to the Schwarzian forms (which are invariant under the Möbious transformation) of the original models. For instance, substituting the truncated Painlevé expansion

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{1}$$

into the Korteweg de Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0, \tag{2}$$

will lead to

$$\sum_{i=0}^{6} X_i \phi^{i-6} = 0, \tag{3}$$

where X_i are functions of $\{u_0,\ u_1,\ u_2\}$ and the derivatives of $\{u_0,\ u_1,\ u_2,\ \phi\}$. To get the Schwarzian form of the KdV equation, one can simply solve the equations

$$X_i = 0,\ i = 0,\ 1,\ ...,\ 6. \tag{4}$$

For the KdV equation, first three equations ($i = 0,\ 1,\ 2$) of (4) give out the results for $\{u_0,\ u_1,\ u_2\}$ while the fourth equation ($i = 3$) of (4) gives out the Schwarzian form

$$\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \lambda = 0, \tag{5}$$

where λ is an arbitrary constant related to the spectral parameter and $\{\phi;\ x\}$ is the usual Schwarzian derivative

$$\{\phi;\ x\} \equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2} \equiv S^{[x]}. \tag{6}$$

The remained equations ($i = 4, 5, 6$) of (4) are satisfied identically.

It is also known that the truncated Painlevé analysis can be used to find some exact solutions for non-completely integrable models[4]. In [5], it has been pointed out that starting from a Möbious transformation invariant form

$$G(S^{[x_i]},\ C^{[x_j x_k]},\ i,\ j,\ k = 1, 2, 3...) = 0, \qquad C^{[x_j x_k]} \equiv \frac{\phi_{x_j}}{\phi_{x_k}}, \tag{7}$$

where G is an arbitrary function of the indicated conformal invariant quantities, one may obtain various integrable models by selecting the function G appropriately. Some concrete forms of G have been given in [10].

It is natural that if one can find out more conformal invariant quantities, then one can include these new conformal invariant quantities into the function G of (7). So the problem is now how to find some more explicit independent conformal invariants. In this section we use simply the truncated Painlevé analysis to find some new conformal invariant quantities especially in high dimensions.

As in the KdV case, for a concrete nonlinear PDE

$$F(x_i,\ u,\ u_{x_i},\ u_{x_i x_j},\ ...) \equiv F(u) = 0, \tag{8}$$

no matter whether it is integrable or not, the truncated Painlevé expansion has the form

$$u = \sum_{i=0}^{\alpha} u_i \phi^{i-\alpha}, \tag{9}$$

where α is a positive integer. Substituting (9) into (8), one may have

$$\sum_{i=0}^{N} X_i \phi^{i-N} = 0, \qquad N > \alpha, \tag{10}$$

where X_i are functions of $\{u_0,\ u_1,\ ...,\ u_\alpha\}$ and the derivatives of $\{u_0,\ u_1,\ ...,\ u_\alpha, \phi\}$. Now to find out possible conformal invariant quantities, we should solve the equations

$$X_i = 0,\ i = 0,\ 1,\ ...,\ N. \tag{11}$$

$\alpha + 1$ equations of (11) should be used to solve out $\{u_0,\ u_1,\ ...,\ u_\alpha\}$. Then substituting the results into the remained equations may yield some of the conformal invariant equations and one may find some new conformal invariants.

For concretely, we use the two dimensional elliptic ϕ^4 model as a simple example

$$u_{xx} + u_{yy} = \sigma u + \mu u^3. \tag{12}$$

Substituting (9) into (12), one can easily find that $\alpha = 1$ and $N = 3$. The first two equations $(i = 0,\ 1)$ of (11) fix u_0 and u_1 as

$$u_0 = \pm\sqrt{\frac{2}{\mu}(\phi_x^2 + \phi_y^2)}, \tag{13}$$

$$u_1 = \pm\frac{1}{6}\sqrt{\frac{2}{\mu}}(\phi_x^2 + \phi_y^2)^{-3/2}(\phi_x^2(\phi_{xx} + \phi_{yy}) + 4\phi_x\phi_y\phi_{xy} + f_y^2(3f_{yy} + f_{xx})). \tag{14}$$

Substituting (13) and (14) into the third equation $(i = 2)$ of (11) yields a Schwarzian form

$$\begin{aligned}
&2C^{[yx]}(7(C^{[yx]})^2 + 9)(C_x^{[yx]})^2 + 2((C^{[yx]})^2 + 3)(3(C^{[yx]})^2 + 1)C_x^{[yx]}C_y^{[yx]} \\
&+C^{[yx]}(5 - 9(C^{[yx]})^4)(C_y^{[yx]})^2 - 6C^{[yx]}(1 + (C^{[yx]})^2)^3(\sigma - S^{[yx]}) \\
&+6C^{[yx]}(1 + (C^{[yx]})^2)^2((C^{[yx]})^2 S^{[xy]} + C^{[yx]}C_{yy}^{[yx]} + S^{[x]}) = 0
\end{aligned} \tag{15}$$

which is invariant under the Möbious transformation, where

$$S^{[xy]} \equiv \frac{\phi_{xxy}}{\phi_y} - \frac{\phi_{xx}\phi_{xy}}{\phi_x\phi_y} - \frac{1}{2}\frac{\phi_{xy}^2}{\phi_y^2}. \tag{16}$$

The final remained equation $(i = 3)$ of (11) is not a Möbious transformation invariant equation.

From Eq. (15) with (16), we obtain not only the conformal invariant quantities $C^{[yx]}$ and $S^{[x]}$ but also two more two dimensional conformal invariant quantities $S^{[xy]}$ and $S^{[yx]}$ which can also be found in the truncated Painlevé analysis of other two dimensional models like the sine-Gordon equation[13]. When $y = x$, $S^{[xy]}$ and $S^{[yx]}$ are both reduced back to the usual Schwarzian derivative $S^{[x]}$.

To find out possible three dimensional Schwarzian derivatives by using the truncated Painlevé analysis, we should start from some three dimensional nonlinear PDEs. The following artificial three dimensional model

$$u_{xyz} - u^4 = 0 \tag{17}$$

may be a simplest example to find a three dimensional Schwarzian derivative.

Substituting (9) into (17), one can easily find that $\alpha = 1$ and $N = 4$. u_0 and u_1 are fixed by the first two equations of (11):

$$u_0 = -(6\phi_x\phi_y\phi_z)^{1/3}, \tag{18}$$

$$\begin{aligned} u_1 = {} & \frac{6^{1/3}}{36(\phi_x\phi_y\phi_z)^{5/3}}(5\phi_z\phi_y^2\phi_x\phi_{xz} + 5\phi_x\phi_y\phi_z^2\phi_{xy} + 5\phi_y\phi_z\phi_x^2\phi_{yz} \\ & +\phi_y^2\phi_z^2\phi_{xx} + \phi_y^2\phi_x^2\phi_{zz} + \phi_x^2\phi_z^2\phi_{yy}). \end{aligned} \tag{19}$$

Substituting (18) and (19) into the third equation of (11) yields a Möbious transformation invariant equation

$$\begin{aligned} & C^{[yx]}\left[12\left(S^{[xy]} + S^{[xz]}\right)\left(C^{[zx]}\right)^4 + \left(12S^{[zx]} + 12S^{[zx]} - 57\left(C_x^{[zx]}\right)^2\right)\left(C^{[zx]}\right)^2 \right. \\ & \left. -14C_x^{[zx]}C_z^{[zx]}C^{[zx]} - \left(C_z^{[zx]}\right)^2\right]\left(C^{[yx]}\right)^4 + \left[36\left(2S^{[xyz]} - C_x^{[zx]}C_x^{[yx]}\right)\left(C^{[zx]}\right)^3 \right. \\ & \left. + \left(12C_z^{[zx]}C_x^{[yx]} - 50C_z^{[yx]}C_x^{[zx]}\right)\left(C^{[zx]}\right)^2 - 6C^{[zx]}C_z^{[zx]}C_z^{[yx]}\right]\left(C^{[yx]}\right)^3 + \left[\left(12S^{[yz]} \right.\right. \\ & \left. -16\left(C_x^{[yx]}\right)^2 + 12S^{[yx]}\right)\left(C^{[zx]}\right)^4 + \left(-32C_z^{[yx]}C_x^{[yx]} + 6C_y^{[yx]}C_x^{[zx]}\right)\left(C^{[zx]}\right)^3 \\ & \left. + \left(2C_z^{[zx]}C_y^{[yx]} - 9\left(C_z^{[yx]}\right)^2\right)\left(C^{[zx]}\right)^2\right]\left(C^{[yx]}\right)^2 + \left(-6\left(C^{[zx]}\right)^3 C_z^{[yx]}C_y^{[yx]} \right. \\ & \left. -8\left(C^{[zx]}\right)^4 C_y^{[yx]}C_x^{[yx]}\right)C^{[yx]} - \left(C^{[zx]}\right)^4\left(C_y^{[yx]}\right)^2 = 0, \end{aligned} \tag{20}$$

where

$$S^{[xyz]} \equiv \frac{\phi_{xyz}}{\phi_x} - \frac{\phi_{xx}\phi_{yz}}{\phi_x^2} - \frac{\phi_{xx}\phi_y\phi_{xz}}{\phi_x^3} - \frac{\phi_{xx}\phi_z\phi_{xy}}{\phi_x^3} + \frac{3}{2}\frac{\phi_{xx}^2\phi_z\phi_y}{\phi_x^4} \tag{21}$$

is a new three dimensional Schwarzian derivative. It is obvious that when $z = y = x$, the three dimensional Schwarzian derivative $S^{[xyz]}$ shown by (21) is reduced back to the usual one dimensional Schwarzian derivative.

In principle, the same procedure can be proceeded further to find higher order higher dimensional Schwarzian derivatives. For instance, use the truncated Painlevé analysis to the following equation

$$u_{xyzt} = u^3, \tag{22}$$

we get a complicated four dimensional fourth order Schwarzian derivative which is a generalization of the usual fourth order one dimensional Schwarzian derivative[15]. It is known that in one dimensional case, higher order Schwarzian derivative can be calculated out from lower order Schwarzian derivative $S^{[x]}$ [15]. The similar situation occurs for high dimensional high order Schwarzian derivatives. The four dimensional fourth order Schwarzian derivative obtained from the truncated Painlevé expansion of (22) can also be expressed by the lower dimensional lower order Schwarzian derivatives $S^{[xyz]}$, $S^{[xy]}$, $S^{[xz]}$, $S^{[yz]}$, $S^{[x]}$ and $C^{[xy]}$ etc. So here we do not write down any complicated higher order higher dimensional Schwarzian derivatives.

3 New high dimensional Painlevé integrable Schwarzian equations

In this section we are interested in whether the obtained new high dimensional Schwarzian derivatives can be used to construct new high dimensional integrable models. It is quite fortunate for us to obtain various new higher dimensional integrable models by using the high dimensional Schwarzian derivatives obtained from the last section. Here we list only some simple special examples on the Schwarzian KdV type extensions.

3.1 (2+1)-dimensional Schwarzian KdV equation with two dimensional Schwarzian derivatives

Usually, it is difficult to obtain isotropic higher dimensional integrable extension(s) of a known lower dimensional integrable model. However, using the conformal invariants, to find some isotropic higher dimensional integrable extensions becomes a straightforward work. For instance,

$$C^{[tx]} + a_1 C^{[ty]} + a_2 S^{[x]} + a_3 S^{[y]} + a_4 S^{[xy]} + a_5 S^{[yx]} = 0, \tag{23}$$

is obviously a (2+1)-dimensional isotropic extension of (1+1)-dimensional Schwarzian KdV equation.

By using the standard Painlevé analysis, we know that Eq. (23) possesses one expansion around the arbitrary non-characteristic manifold

$$\phi = \sum_{i=0}^{\infty} \phi_i \psi^{i-1} \tag{24}$$

with arbitrary functions $\{\phi_0,\ \phi_1,\ \psi\}$ and the expansions around arbitrary characteristic manifold

$$\phi = f_0(x_1,\ t) + \sum_{i=0}^{\infty} \phi_i(x_1,\ t)\,(x_2 + \psi(x_1,\ t))^{i+1}\,, \tag{25}$$

for arbitrary $\{f_0(x_1,\ t),\ \phi_0(x_1,\ t),\ \phi_1(x_1,\ t),\ \psi(x_1,\ t)\}$, and two

$$\phi = f_0(x_1,\ t) + \sum_{i=0}^{\infty} \phi_i(x_1,\ t)(x_2 + \psi(x_1,\ t))^{i+3}, \tag{26}$$

for arbitrary $f_0(x_1,\ t)$ and $\psi(x_1,\ t)$, where $\{x_1,\ x_2\} = \{x,\ y\}$ or $\{x_1,\ x_2\} = \{y,\ x\}$. From the expansions (24)–(26), we know that all the solutions of the (2+1)-dimensional Schwarzian KdV equation are single valued about arbitrary non-characteristic and characteristic manifolds and then the model is completely integrable[14].

3.2 (3+1)-dimensional Schwarzian KdV equation with three dimensional Schwarzian derivative

Using the three dimensional Schwarzian derivative $S^{[xyz]}$, we can obtain the simplest (3+1)-dimensional Schwarzian KdV equation

$$C^{[tx]} + S^{[xyz]} = 0 \tag{27}$$

which can be reduced back to the usual (1+1)-dimensional Schwarzian KdV equation obviously.

Using the standard Painlevé analysis one can easily prove that the (3+1)-dimensional Schwarzian KdV equation (27) is Painlevé integrable. Actually, the Schwarzian KdV equation (27) possesses the non-characteristic singularities of the form

$$\phi = \sum_{i=0}^{\infty} \phi_i \psi^{i-1} \tag{28}$$

with arbitrary $\phi_0,\ \phi_1,\ \psi$ and the characteristic singularities of the forms

$$\phi = f_0(y,z,t) + \sum_{i=0}^{\infty} \phi_i(y,z,t)(x+\psi(y,z,t))^{i+1}, \qquad f_0,\ \phi_0,\ \phi_1,\ \psi \text{ arbitrary}, \tag{29}$$

$$\phi = f_0(x,y,t) + \sum_{i=0}^{\infty} \phi_i(x,y,t)(z+\psi(x,y,t))^{i+1}, \qquad f_0,\ \phi_0,\ \phi_1,\ \psi \text{ arbitrary}, \tag{30}$$

$$\phi = f_0(y,z,t) + \sum_{i=0}^{\infty} \phi_i(y,z,t)(z+\psi(y,z,t))^{i+3}, \qquad f_0,\ \psi \text{ arbitrary}, \tag{31}$$

and

$$\phi = f_0(x,y,t) + \sum_{i=0}^{\infty} \phi_i(x,y,t)(z+\psi(x,y,t))^{i+3}, \qquad f_0,\ \psi \text{ arbitrary} \tag{32}$$

which means all the solutions of (27) are single valued about arbitrary manifold no matter whether the manifold is non-characteristic or characteristic.

Furthermore, one can prove that the generalized high dimensional Schwarzian KdV type model

$$\sum_{i=0}^{N} a_i S^{[x_i]} + \sum_{i=0}^{N}\sum_{j=0}^{N} b_{ij} S_{[x_i x_j]} + \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=1}^{N} c_{ijk} S_{[x_i x_j x_k]} + g(C^{[x_i x_j]}) = 0, \tag{33}$$

for some suitable polynomial functions of $C^{[x_i x_j]}$, $g(C^{[x_i x_j]})$, are Painlevé integrable.

4 Summary and discussions

In summary, applying the truncated Painlevé analysis to high dimensional high order PDEs, one may obtain some types of high dimensional high order Schwarzian derivatives. Especially, an explicit three dimensional Schwarzian derivative is derived from a three dimensional PDE

while n ($n \geq 4$) dimensional m ($m \geq 4$) order Schwarzian derivatives which are invariant under the Möbious transformation can be expressed by means of the lower dimensional lower order Schwarzian derivatives.

Using two dimensional and three dimensional Schwarzian derivatives, we may obtain various new Painlevé integrable models. Some of high dimensional Painlevé integrable KdV type Schwarzian equations are given.

Though the Painlevé property is considered as a sufficient condition of the integrability, and many of other integrable properties like the Lax pair, symmetries, Bäcklund-Darboux transformation and multi-soliton solutions can be obtained from the usual Painlevé analysis for (1+1)- and (2+1)-dimensional models, it is still open how to obtain other integrable properties from the Painlevé analysis in higher dimensions especially in (3+1)-dimensions. The more about the high dimensional Schwarzian derivatives and the related high dimensional integrable models is worthy of studying further.

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